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ORIGINAL PAPER

Long time behaviour of a stochastic model for continuous flow bioreactor

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Abstract The global dynamics of a deterministic model in wastewater treatment has been investigated in Zhang (J Math Chem 50:2239–2247, 2012). The stochastic version, which can be used for continuous flow bioreactor and membrane reactor is presented in this study. Precisely, we assume there is some uncertainty in the part describing the recycle, which results in a set of stochastic differential equations with white noise. We first show that the stochastic model has always a unique positive solution. Then long time behavior of the model is studied. Our study shows that both the washout equilibrium and non-washout equilibrium are stochastically stable. At the end, we carry out some numerical simulations, which support our theoretical conclusions well.

Keywords Bioreactor · Stochastic model · Deterministic model · Dynamics · Stochastic stability · Equilibrium

1 Introduction

Industrial effluent and household sewage are complex mixtures of many substrates and microorganisms. The purpose of wastewater treatment is to remove pollutants what can harm the aquatic environment. Let $S(t)$ be the concentration of substrate and $X(t)$ be the concentration of microorganism at time t, respectively. Then after non-dimensionalization, a mathematical model for wastewater treatment process can have the following form.

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$$
\begin{cases} \frac{dS}{dt} = \frac{1}{\tau}(S_0 - S) - \frac{SX}{1+S} - m_S X, \\ \frac{dX}{dt} = \beta \frac{1}{\tau}(X_0 - X) + \gamma \frac{R}{\tau} X + \frac{SX}{1+S} - k_d X, \end{cases} \tag{1.1}
$$

which actually includes two types of models: (a) continuous flow reactors used in treatment of industrial wastewater; (b) membrane reactors designed for domestic wastewater treatment. It has been investigated by Nelson and Zhang [11,14]. It shows that if $X_0 = 0$, namely there is no microorganism in the influent, then from the analysis of [11], the dynamics of a reactor model with idealized recycle is equivalent to the idealized membrane reactor, and that of a reactor model with non-idealized recycle is equivalent to a non-idealized membrane reactor model. In other words, the case of $\beta = \gamma = R = 1$ is equivalent to the case of $\beta = \gamma = 0$ and the case of $0 < \beta < 1, \gamma = 0$ is equivalent to the case of $\beta = \gamma = 1, 0 < R < 1$. Therefore, in this study, we shall set our parameters as follows, $\beta = \gamma = 1, 0 \leq R \leq 1$ and $X_0 = 0$, $S_0 \neq 0$, where $R = 0$, $0 < R < 1$ and $R = 1$ correspond models of a continuous flow reactor without recycle, a continuous flow reactor with non-idealized recycle and a continuous flow reactor with idealized recycle, respectively. Previous results also indicate that model (1.1) has at most two equilibria: a washout equilibrium $E_1(S_0, 0)$ and a non-washout equilibrium $E_2(S^*, X^*)$ where

$$
S^* = \frac{1 - R + k_d \tau}{R - 1 + (1 - k_d)\tau}, X^* = \frac{S_0 - S^*}{1 - R + (k_d + m_s)\tau}.
$$

To have E_2 physically meaningful, namely both S^*, X^* should be positive, one of the following conditions should be satisfied.

(i) $k_d < \frac{S_0}{1+S_0}$ and $\tau \geq \frac{(1-R)(1+S_0)}{S_0-(1+S_0)k_d}$ for $0 \leq R < 1$;
(ii) $k_d < \frac{S_0}{1+S_0}$ for $R = 1$.

Then from [11,14] we know that both E_1 and E_2 are stable when they exist.

Many researchers' works are based on deterministic models since they are useful and able to provide reasonably good predications to what they describe in physical and engineering processes. For example, the authors of [3,4,11–13] investigated the deterministic models for bioreactor. However, in reality, uncertainties always exist. Very often these uncertainties are ignored or their study are delayed or omitted. This limits our understanding of the true processes. Recently, lots of works in the direction of stochastic modeling have been carried out and being developed very quickly due to the needs in the related research areas. To name but a few, [1,2,5–10,15] and the references therein. In practice, there might be different ways through which uncertainties can have influences on the physical or engineering processes. Such processes with uncertainties can be modeled by stochastic equations, which play an important role in describing the dynamical models in biological, medical, physical and social sciences.

In this paper, we consider the case of uncertainties from the recycle, namely the recycle rate takes the form as

$$
R \to R + \sigma dB(t)
$$

in system (1.1), where $B(t)$ is a scalar Brownian motion. Then we are ready to propose our model as follows, which is a stochastic model in Itô form.

$$\begin{cases} \frac{dS}{dt} = \frac{1}{\tau}(S_0 - S) - \frac{SX}{1+S} - m_S X, \\ \frac{dX}{dt} = \beta\frac{1}{\tau}(X_0 - X) + \gamma\frac{R}{\tau}X + \frac{SX}{1+S} - k_d X + \gamma\frac{\sigma X}{\tau}\frac{dB(t)}{dt}. \end{cases} \tag{1.2}$$

We are interested in the effect of uncertainties on the dynamics of the above model. As the model describes a bioreactor and $S(t)$ and $X(t)$ are concentrations, which can not be negative, of the substrate and microorganism, respectively, we would like to find out whether or not the solution of the stochastic model (1.2)

- will remain positive or nonnegative,
- will not explode to infinity in a finite time,
- will be persistent.

The rest of the paper is organised as follows. In Sect. 2, we show that there is a unique nonnegative solution no matter how large the intensities of noise is, and the solution will not explode in a finite time. By the Lyapunov function constructed, Sect. 3 shows if the noise satisfied certain condition, then the washout equilibrium E_1 is stochastically asymptotically stable. In Sect. 4, we show that how the solution goes around the non-washout equilibrium, E_2 of the deterministic model under different conditions, which implies that the solution is persistent and tends to the equilibrium state if the intensity is not very strong. Section 5 concludes the paper with numerical simulations.

2 Existence and uniqueness of the positive solution

In this section, for model (1.2), we claim that there is a unique nonnegative solution no matter how large the intensities of noise is, which is summarised in the following theorem.

Theorem 2.1 *There is a unique solution* $(S(t), X(t))$ *of model* (1.2) *on* $[0, \infty)$ *for any initial value* $(S(0), X(0)) \in R^2_+ = \{x \in R^2 : x_i > 0, i = 1, 2\}$*, and the solution will remain in* R^2_+ *with probability 1, namely* $(S(t), X(t) \in R^2_+$ *for all* $t \geq 0$ *almost surely.*

Proof Since the coefficients of model (1.2) are locally Lipschitz continuous for any given initial value $(S(0), X(0)) \in R^2_+$, there is a unique local solution $(S(t), X(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show this solution is global, we need to show that $\tau_e = \infty$ *a.s.*, which is the abbreviation for "almost surely". Let $m_0 \geq 0$ be sufficiently large so that $S(0) \in [\frac{1}{m_0}, m_0]$, $X(0) \in [\frac{1}{m_0}, m_0]$. For each integer $m \geq m_0$, define the stopping time

$$\tau_m = \inf\left\{t \in [0, \tau_e) : S(t) \notin \left(\frac{1}{m}, m\right) \text{ or } X(t) \notin \left(\frac{1}{m}, m\right)\right\},$$

where throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_m is increasing as $m \to \infty$. Setting $\tau_\infty = \lim_{m\to\infty} \tau_m$ yields $\tau_\infty \leq \tau_e$ *a.s.*.

If we can show that $\tau_\infty = \infty$ *a.s.*, then $\tau_e = \infty$ and $(S(t), X(t) \in R_+^2$ for all $t \geq 0$ *a.s.*. In other words, to complete the proof, all we need to show is that $\tau_\infty = \infty$ *a.s.*. If this statement is false, then there is a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$P\{\tau_\infty \leq T\} > \varepsilon.$$

Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon \tag{2.1}$$

for all $m \geq m_1$. Define a C^2-function V: $R_+^2 \to \bar{R}_+$,

$$V(S(t), X(t)) = \left(S - c - c\ln\frac{S}{c}\right) + (X - 1 - \ln X),$$

where c is a positive constant to be determined. Then $V(S, X) \geq 0$ since $u - 1 - \ln u \geq 0, \forall u > 0$. Using Itô's formula, we get

$$\begin{aligned} dV &= dS - \frac{c}{S}dS + \frac{c}{2S^2}(dS)^2 + dX - \frac{1}{X}dX + \frac{1}{2X^2}(dX)^2 \\ &= LVdt + \left(\frac{\sigma X}{\tau} - \frac{\sigma}{\tau}\right)dB(t), \end{aligned}$$

where

$$\begin{aligned} LV = &\frac{1}{\tau}(S_0 - S) - \frac{SX}{1+S} - m_S X - \frac{c}{S\tau}(S_0 - S) + \frac{cX}{1+S} + m_S c\frac{X}{S} - \frac{1}{\tau}X + \frac{R}{\tau}X \\ &+ \frac{SX}{1+S} - k_d X + \frac{1}{\tau} - \frac{R}{\tau} - \frac{S}{1+S} + k_d + \frac{\sigma^2}{2\tau^2}. \end{aligned}$$

Note that $R \in [0, 1]$ is the recycle rate, and $S(t)$ and $X(t)$ are real numbers satisfied $0 < S(t) < \infty$, $0 < X(t) < \infty$. We choose $c = k_d$, then

$$LV \leq \frac{1}{\tau}S_0 + \frac{c}{\tau} + m_S c K_1 + \frac{1}{\tau} - \frac{R}{\tau} + k_d + \frac{\sigma^2}{2\tau^2} \stackrel{\triangle}{=} K.$$

Therefore,

$$\int_0^{\tau_m \wedge T} dV(S(r), X(r)) \leq \int_0^{\tau_m \wedge T} K dr + \int_0^{\tau_m \wedge T} \left(\frac{\sigma X}{\tau} - \frac{\sigma}{\tau}\right) dB(r).$$

Taking expectation on each side of the above inequality yields

$$E[V(S(\tau_m \wedge T), X(\tau_m \wedge T))] \leq V(S(0), X(0)) + E\int_0^{\tau_m \wedge T} K dr \leq V(S(0), X(0)) + KT. \quad (2.2)$$

Let $\Omega_m = \{\tau_m \leq T\}$ for $m \geq m_1$. From (2.1), we have $P(\Omega_m) \geq \varepsilon$. Note that for every $\omega \in \Omega_m$, there is at least one of $S(\tau_m, \omega)$, $X(\tau_m, \omega)$ equals either m or $\frac{1}{m}$. If $S(\tau_m, \omega) = m$ or $\frac{1}{m}$, then

$$V(S(\tau_m \wedge T), X(\tau_m \wedge T)) \geq (m - c - c\ln\frac{m}{c}) \wedge \left(\frac{1}{m} - c - c\ln\frac{1}{mc}\right),$$

while if $X(\tau_m, \omega) = m$ or $\frac{1}{m}$, then

$$V(S(\tau_m \wedge T), X(\tau_m \wedge T)) \geq (m - 1 - \ln m) \wedge \left(\frac{1}{m} - 1 - \ln\frac{1}{m}\right).$$

Consequently,

$$\begin{aligned} &V(S(\tau_m \wedge T), X(\tau_m \wedge T)) \\ &\quad \geq \left(m - c - c\ln\frac{m}{c}\right) \wedge \left(\frac{1}{m} - c - c\ln\frac{1}{mc}\right) \\ &\quad \wedge (m - 1 - \ln m) \wedge \left(\frac{1}{m} - 1 - \ln\frac{1}{m}\right). \end{aligned}$$

It then follows from equations (2.1) and (2.2) that

$$\begin{aligned} &V(S(0), X(0)) + KT \\ &\quad \geq E[1_{\Omega_m(\omega)} V(S(\tau_m \wedge T), X(\tau_m \wedge T))] \\ &\quad \geq (m - c - c\ln\frac{m}{c}) \wedge \left(\frac{1}{m} - c - c\ln\frac{1}{mc}\right) \\ &\quad \wedge (m - 1 - \ln m) \wedge \left(\frac{1}{m} - 1 - \ln\frac{1}{m}\right), \end{aligned}$$

where $1_{\Omega_m(\omega)}$ is the indicator function of Ω_m. Letting $m \to \infty$ leads to the contradiction $\infty > V(S(0), X(0)) + KT = \infty$. It implies that $\tau_\infty \leq \tau_e$ $a.s.$. This completes the proof. $\square$

3 Stochastically asymptotically stable of the washout equilibrium of the stochastic model

In the case of deterministic model, we know the washout equilibrium $E_1(S_0, 0)$ always stable [14], which means the microorganisms will die out after some period of time. It is easy to check that it is still the equilibrium of the stochastic model (1.2). Now, in this section we shall investigate the stability of E_1 when the model with uncertainty. In other words, we are interested in the effect of uncertainty on the stability of E_1. We have the following conclusion.

Theorem 3.1 *When the intensity of the noise, σ satisfies*

$$\sigma^2 \leq 2\tau - 2\tau R - 2\frac{S_0}{1+S_0}\tau + 2k_d\tau,$$

the washout equilibrium $E_1(S_0, 0)$ of model (1.2) is stochastically asymptotically stable.

Proof Let $(S(t), X(t))$ be the solution of model (1.2) with initial value $(S(0), X(0) \in R_+^2$. Letting $X_1 = S - S_0,\ X_2 = X$ gives

$$\begin{cases} dX_1 = \left(-\frac{1}{\tau}X_1 + \frac{(X_1+S_0)X_2}{1+X_1+S_0} - m_s X_2\right)dt, \\ dX_2 = \left(-\frac{1}{\tau}X_2 + \frac{R}{\tau}X_2 + \frac{(X_1+S_0)X_2}{1+X_1+S_0} - k_d X_2\right)dt + \frac{\sigma}{\tau}X_2 dB(t). \end{cases} \tag{3.1}$$

Linearise (3.1) at $(0, 0)$. Then we get

$$\begin{cases} dX_1 = \left(-\frac{1}{\tau}X_1 + \frac{S_0X_2}{1+S_0} - m_s X_2\right)dt, \\ dX_2 = \left(-\frac{1}{\tau}X_2 + \frac{R}{\tau}X_2 + \frac{S_0X_2}{1+S_0} - k_d X_2\right)dt + \frac{\sigma}{\tau}X_2 dB(t). \end{cases} \tag{3.2}$$

Next we show the equilibrium $(X_1, X_2) = (0, 0)$ of model (3.2) is stochastically asymptotically stable. To this end, define a C^2-function V given by

$$V = X_1^2 + X_2^2 + AX_2,$$

where A is a positive constant. Obviously, V is positive definite, and along the trajectories of model (3.2) we have

$$\begin{aligned} dV &= 2X_1dX_1 + (dX_1)^2 + 2X_2dX_2 + (dX_2)^2 + AdX_2 \\ &= LVdt + \left(\frac{2\sigma X_2^2}{\tau} + \frac{A\sigma X_2}{\tau}\right)dB(t), \end{aligned}$$

where

$$LV = -\frac{2}{\tau}X_1^2 - \frac{2S_0}{1+S_0}X_1X_2 - 2m_sX_1X_2 - \frac{2}{\tau}X_2^2 + \frac{2R}{\tau}X_2^2 + \frac{2S_0}{1+S_0}X_2^2 - 2k_dX_2^2 + \frac{\sigma^2}{\tau^2}X_2^2 + A\left(-\frac{1}{\tau} + \frac{R}{\tau} + \frac{S_0}{1+S_0} - k_d\right)X_2.$$

We claim that LV is a negative function under certain condition. In fact,

(i) If $X_1 \geq 0$, then

$$LV \leq -\frac{2}{\tau}X_1^2 + \left(\frac{2R}{\tau} - \frac{2}{\tau} + \frac{2S_0}{1+S_0} - 2k_d + \frac{\sigma^2}{\tau^2}\right)X_2^2.$$

Obviously when $\sigma^2 \leq 2\tau - 2\tau R - 2\frac{S_0}{1+S_0}\tau + 2k_d\tau$, $LV \leq 0$, and $LV = 0$ if and only if $X_1 = X_2 = 0$.

(ii) If $X_1 < 0$, we can get $-X_1 < S_0$ from $S = X_1 + S_0 > 0$, then

$$LV \leq -\frac{2}{\tau}X_1^2 + \frac{2S_0^2}{1+S_0}X_2 + 2m_sS_0X_2 - \frac{2}{\tau}X_2^2 + \frac{2R}{\tau}X_2^2 + \frac{2S_0}{1+S_0}X_2^2 - 2k_dX_2^2 + \frac{\sigma^2}{\tau^2}X_2^2 + A\left(-\frac{1}{\tau} + \frac{R}{\tau} + \frac{S_0}{1+S_0} - k_d\right)X_2.$$

We then can choose

$$A = \frac{\frac{2S_0^2}{1+S_0} + 2m_sS_0}{\frac{1}{\tau} - \frac{R}{\tau} - \frac{S_0}{1+S_0} + k_d},$$

which yields

$$LV \leq -\frac{2}{\tau}X_1^2 + \left(\frac{2R}{\tau} - \frac{2}{\tau} + \frac{2S_0}{1+S_0} - 2k_d + \frac{\sigma^2}{\tau^2}\right)X_2^2.$$

Again, when $\sigma^2 \leq 2\tau - 2\tau R - 2\frac{S_0}{1+S_0}\tau + 2k_d\tau$, $LV \leq 0$, and $LV = 0$ if and only if $X_1 = X_2 = 0$. Therefore the (0, 0) origin model (3.2) is stochastically asymptotically stable.

Next, we show the origin (0,0) of model (3.1) is stochastically asymptotically stable, namely the washout equilibrium $E_1(S_0, 0)$ of model (1.2) is stochastically asymptotically stable. Notice

$$
\begin{aligned}
&|f(t,X)-F\cdot X|+|g(t,X)-G\cdot X|\\
&=\sqrt{\left(\frac{S_0}{1+S_0}X_2-\frac{(X_1+S_0)X_2}{1+X_1+S_0}\right)^2+\left(\frac{(X_1+S_0)X_2}{1+X_1+S_0}-\frac{S_0}{1+S_0}X_2\right)^2}\\
&=\sqrt{2\left(\frac{S_0}{1+S_0}X_2-\frac{(X_1+S_0)X_2}{1+X_1+S_0}\right)^2}\\
&=2\left|\frac{X_1X_2}{(1+X_1+S_0)(1+S_0)}\right|.
\end{aligned}
$$

Then by [7] and [9], for small $\varepsilon > 0$, when $|X_1| < \varepsilon,\ |X_2| < \varepsilon$

$$
\begin{aligned}
&|f(t,X)-F\cdot X|+|g(t,X)-G\cdot X|\\
&\le 2\varepsilon\left|\frac{X_1}{(1+X_1+S_0)(1+S_0)}\right|\\
&\le 2\varepsilon\left|\frac{X_1}{1+S_0}\right|\\
&\le l\varepsilon
\end{aligned}
$$

where $l = \frac{2}{1+S_0}\varepsilon$. Therefore the washout equilibrium $E_1(S_0, 0)$ of model (1.2) is stochastically asymptotically stable. □

4 Asymptotic behavior around the non-washout equilibrium of the deterministic model

When studying the deterministic model (1.1), we are interested in two things. One is when the microorganism will die out, which has been shown in Sect. 3. Another is the persistence of the microorganism. In the case of deterministic model, the second problem is solved by showing that the non-washout equilibrium is a global attractor or is globally asymptotically stable. Generally, there is no non-washout equilibrium for stochastic model (1.2). Since model (1.2) can be treated as the perturbation of model (1.1) which has an non-washout equilibrium $E_2(S^*, X^*)$, it is reasonable to consider the microorganism will persist if the solution of model (1.2) is oscillating around $E_2(S^*, X^*)$ most of the time. In this sense, we have conclusion as follows.

Theorem 4.1 *Let* $(S(t), X(t))$ *be a solution of model* (1.2) *with initial value* $(S(0), X(0)) \in R^2_+$. *Then when* $\sigma^2 \le \min\{\frac{\tau}{2}, \tau - \frac{\tau R}{2} + \frac{k_d\tau}{2} + \frac{m_s\tau}{2}\}$,

$$
\lim_{t\to\infty}\sup\frac{1}{t}E\int_0^t[(S(u)-S^*)^2+r^2(X(u)-X^*)^2]du \le k_\sigma,
$$

where

$$r^2 = \frac{\frac{4\sigma^2}{\tau^2} - \frac{2}{\tau} + \frac{2R}{\tau} - 2k_d - 2m_s}{\frac{4\sigma^2}{\tau^2} - \frac{2}{\tau}},$$
$$k_\sigma = \frac{c}{\frac{2}{\tau} - \frac{4\sigma^2}{\tau^2}},$$
$$c = \frac{4\sigma^2}{\tau^2}(R - \tau k_d - \tau m_s)^2 (X^*)^2 + \frac{2\sigma^2}{\tau^2}(S_0)^2 + \left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(1 + S^*)\frac{\sigma^2 X^*}{2\tau^2}.$$

Proof Define a C^2-function V as

$$V = [(S - S^*)^2 + (X - X^*)^2] + \left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(1 + S^*)\left(X - X^* - X^* \ln \frac{X}{X^*}\right),$$

where $\sigma^2 \leq \min\{\frac{\tau}{2}, \tau - \frac{\tau R}{2} + \frac{k_d \tau}{2} + \frac{m_s \tau}{2}\}$. Then V is a positive definite. Using Itô's formula, we get

$$\begin{aligned} dV &= 2[(S - S^*) + (X - X^*)](dS + dX) + (dS)^2 + (dX)^2 \\ &\quad + \left(-\frac{4\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(1 + S^*)\left[\left(1 - \frac{X^*}{X}\right)dX + \frac{X^*}{2X^2}(dX)^2\right] \\ &= LVdt + \left[2(S - S^*) + 2(X - X^*)\frac{\sigma X}{\tau}\right. \\ &\quad \left. + \left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(1 + S^*)\left(1 - \frac{X^*}{X}\right)\frac{\sigma X}{\tau}\right] dB(t), \end{aligned}$$

where

$$\begin{aligned} LV &= [2(S - S^*) + 2(X - X^*)]\left[\frac{1}{\tau}(S_0 - S) - \frac{1}{\tau}X - m_s X + \frac{R}{\tau}X - k_d X\right] \\ &\quad + \frac{\sigma^2 X^2}{\tau^2} + \left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(1 + S^*)(X - X^*) \\ &\quad \times \left(-\frac{1}{\tau} + \frac{R}{\tau} + \frac{S}{1 + S} - k_d\right) \\ &\quad + \left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(1 + S^*)\frac{\sigma^2 X^*}{2\tau^2}. \end{aligned}$$

Note that $\frac{1}{\tau}(S_0 - S^*) - \frac{S^* X^*}{1+S^*} - m_s X^* = 0$ implies $\frac{S^*}{1+S^*} = \frac{1}{\tau} - \frac{R}{\tau} + k_d$. Then we have

$$\begin{aligned}\frac{1}{\tau}S_0 &= \frac{1}{\tau}S^* + \frac{S^*X^*}{1+S^*} + m_s X^* \\ &= \frac{1}{\tau}S^* + \frac{1}{\tau}X^* - \frac{R}{\tau}X^* + k_d X^* + m_s X^*\end{aligned}$$

and

$$\begin{aligned}&[2(S-S^*)+2(X-X^*)]\left[\frac{1}{\tau}S_0 - \frac{1}{\tau}S - \frac{1}{\tau}X + \frac{R}{\tau}X - m_s X - k_d X\right] \\ &= [2(S-S^*)+2(X-X^*)]\left[-\frac{1}{\tau}(S-S^*) + (-\frac{1}{\tau} + \frac{R}{\tau} - m_s - k_d)(X-X^*)\right] \\ &= -\frac{2}{\tau}(S-S^*)^2 + 2\left(-\frac{1}{\tau} + \frac{R}{\tau} - m_s - k_d\right)(X-X^*)^2 \\ &\quad + \left(-\frac{4}{\tau} + \frac{2R}{\tau} - 2m_s - 2k_d\right)(X-X^*)(S-S^*),\end{aligned}$$

which imply the following hold.

$$\begin{aligned}\frac{\sigma^2 X^2}{\tau^2} &\le \frac{\sigma^2}{\tau^2}(X+S-S_0+S_0)^2 \\ &\le 2\frac{\sigma^2}{\tau^2}[(X+S-S_0)^2 + (S_0)^2] \\ &= 2\frac{\sigma^2}{\tau^2}[(X+S-S^*-X^*+RX^*-\tau k_d X^* - \tau m_s X^*)^2 + (S_0)^2] \\ &= 2\frac{\sigma^2}{\tau^2}[((X-X^*)+(S-S^*)+(R-\tau k_d-\tau m_s)^2(X^*)^2 + (S_0)^2] \\ &\le 2\frac{\sigma^2}{\tau^2}[2((X-X^*)+(S-S^*))^2 + 2(R-\tau k_d-\tau m_s)^2(X^*)^2 + (S_0)^2] \\ &= \frac{4\sigma^2}{\tau^2}(X-X^*)^2 + \frac{4\sigma^2}{\tau^2}(S-S^*)^2 + \frac{8\sigma^2}{\tau^2}(X-X^*)(S-S^*) \\ &\quad + \frac{4\sigma^2}{\tau^2}(R-\tau k_d-\tau m_s)^2(X^*)^2 + \frac{2\sigma^2}{\tau^2}(S_0)^2,\end{aligned}$$

and

$$\begin{aligned}&\left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(1+S^*)(X-X^*)\left(-\frac{1}{\tau} + \frac{R}{\tau} + \frac{S}{1+S} - k_d\right) \\ &= \left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(1+S^*)(X-X^*)\left(\frac{S}{1+S} - \frac{S^*}{1+S^*}\right) \\ &= \left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)\frac{(X-X^*)(S-S^*)}{1+S} \\ &\le \left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(X-X^*)(S-S^*).\end{aligned}$$

Therefore

$$LV \leq \left(\frac{4\sigma^2}{\tau^2} - \frac{2}{\tau}\right)(S - S^*)^2 + \left(\frac{4\sigma^2}{\tau^2} - \frac{2}{\tau} + \frac{2R}{\tau} - 2k_d - 2m_s\right)(X - X^*)^2 + c,$$

where

$$\begin{aligned} c = & \frac{4\sigma^2}{\tau^2}(R - \tau k_d - \tau m_s)^2 (X^*)^2 + \frac{2\sigma^2}{\tau^2}(S_0)^2 \\ & + \left(-\frac{8\sigma^2}{\tau^2} + \frac{4}{\tau} - \frac{2R}{\tau} + 2k_d + 2m_s\right)(1 + S^*)\frac{\sigma^2 X^*}{2\tau^2}. \end{aligned}$$

From

$$E\int_0^t dV = E\int_0^t LVdt,$$

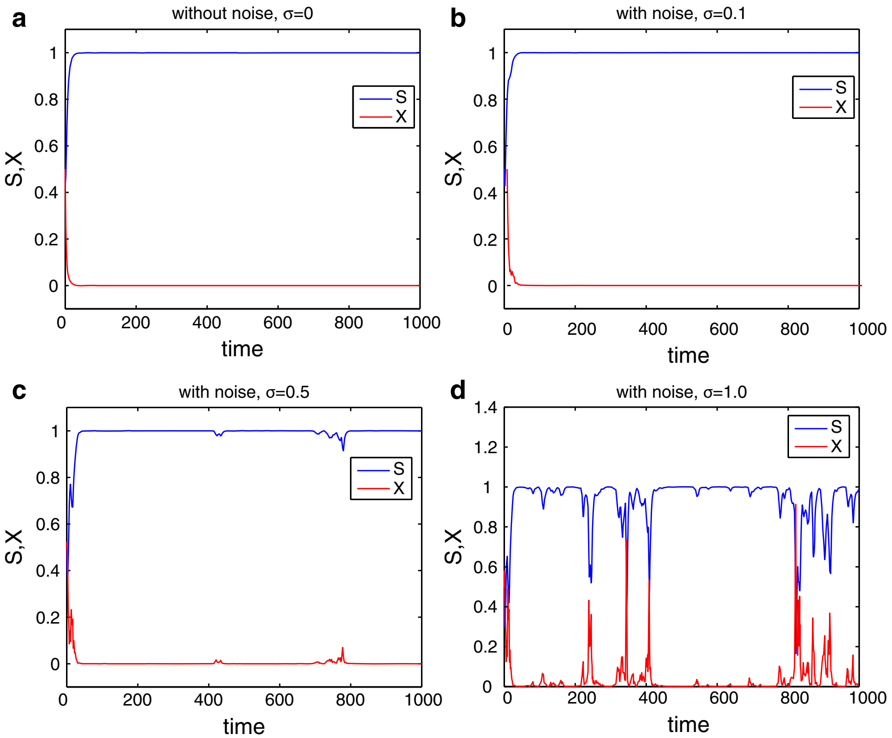

Fig. 1 Comparison of the dynamics in deterministic model and stochastic model with $\sigma = 0.1, 0.5, 1$, respectively

we know

$$\lim_{t\to\infty}\sup \frac{1}{t} E\int_0^t [(S(u)-S^*)^2 + r^2(X(u)-X^*)^2]du \le k_\sigma,$$

where r^2 and k_σ are defined in the theorem statement. This completes the proof. □

Theorem 4.1 and its proof also tell us that the average in time of the second moment of the solution will be bounded.

5 Numerical simulations and conclusion

Numerical simulations are carried out in this section to demonstrate the stochastic stability of E_1 and E_2, and also effect of uncertainty on the dynamics of the model.

First, we illustrate washout equilibrium E_1 is stochastically stable. To this end, we choose (0.5, 0.5) as the initial value of model (1.2), and let $S_0 = 1$, $R = 0.9$, $k_d = 0.6$, $m_S = 0.5$, $\tau = 2$, and σ varies in a range governed by Theorem 3.1. Since $k_d > \frac{S_0}{1+S_0} = 0.5$, $R = 0.9$ and $\sigma^2 \le 2\tau - \tau R - 2\frac{S_0}{1+S_0}\tau + 2k_d\tau = 2.6$, from

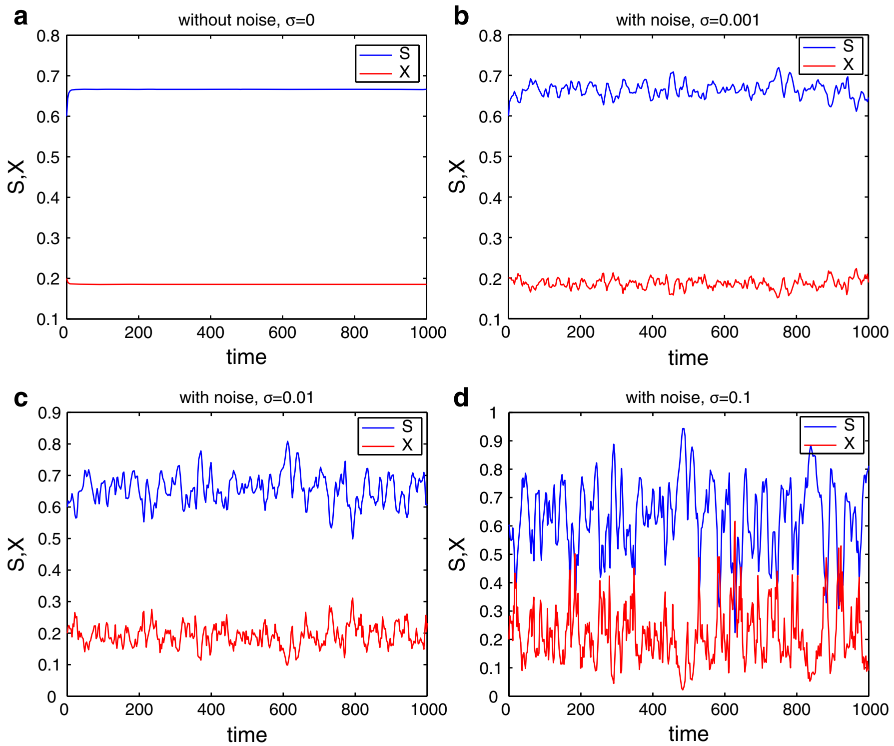

Fig. 2 Comparison of the dynamics in deterministic model and stochastic model with $\sigma = 0.001, 0.01, 0.1$, respectively

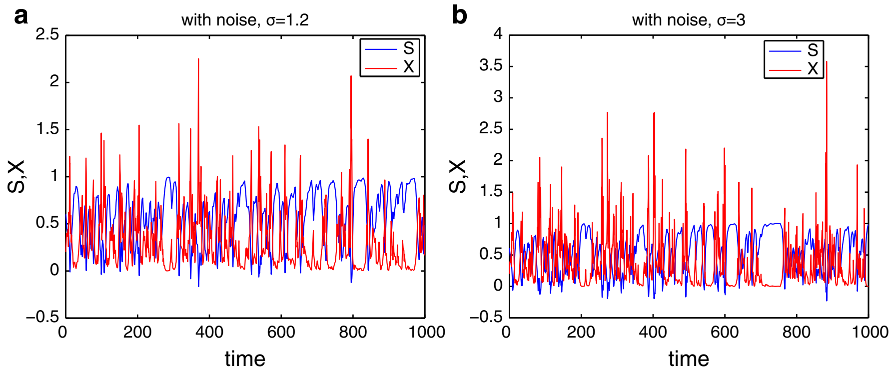

Fig. 3 The long time behavior of the stochastic model with $\sigma = 1.2, 3$, respectively

Sect. 3, the equilibrium $E_1(S_0, 0)$ is globally asymptotically stable. Our simulation also supports this conclusion as shown in Fig. 1, where we show the effect from different intensities, σ of noises. As seen, when σ is relatively small such as less than 0.5, there are not much effect from the noise; when σ is large, but less than 2.6 it is still stable although the effect is obvious. In practice, it implies if the uncertainty can be controlled within certain range, the microorganism will die out eventually, which preserve the dynamics of the deterministic model.

Next we show E_2 is stochastically stable. For this purpose, in our simulation, the parameters are set as $S_0 = 1$, $R = 1$, $k_d = 0.4$, $m_S = 0.5$, $\tau = 2$. Since $k_d < \frac{S_0}{1+S_0} = 0.5$ and $R = 1$, model (1.1) has an unique positive equilibrium $E_2(S^*, X^*) = (\frac{2}{3}, \frac{5}{27})$, which according to result in Sect. 4,when $\sigma^2 \leq \min\{\frac{\tau}{2}, \tau - \frac{\tau R}{2} + \frac{k_d \tau}{2} + \frac{m_s \tau}{2}\} = \min\{1, 1.9\} = 1$, is stable. Our simulation in this case uses (0.66, 0.18) as the initial value. It shows in Fig. 2 that the solution of the model (1.2) oscillates around the point $E_2(S^*, X^*)$ at the most of time, which implies E_2 is stochastically stable when σ satisfies the condition in Theorem 4.1.

While for $\sigma^2 > \min\{\frac{\tau}{2}, \tau - \frac{\tau R}{2} + \frac{k_d \tau}{2} + \frac{m_s \tau}{2}\} = \min\{1, 1.9\} = 1 E_2$ is unstable, please see Fig. 3, which shows the long time behavior of the model (1.2) with $\sigma = 1.2, 3$, respectively.

The implication of Figs. 2 and 3 is that in practice the microorganisms can persist if the noise is controlled in certain level due to the positive equilibrium is stochastically stable.

To sum up, in this paper, we first proposed a stochastic model and then analyzed the long time dynamics of a stochastic model. Then proved the existence of the positive solution. The conditions for globally stochastic stability of the washout and non-washout equilibria have been given. At the end, numerical simulations have also been carried out, which support our theoretical analysis in previous sections.

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